Math Appreciation – Connections in Context
ECUR 992 Project

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Introduction

Positionality is a reflex strategy that identifies a researcher's bias (McMillan, 2010). As a teacher of mathematics in western Canada, I want to begin by identifying my bias. My philosophy in mathematics education is to develop students' mathematical thinking. Too often I lose this focus and concentrate on transmitting algorithms and procedures for students to follow. In completing a project for my masters, I wanted to explore principles and practices in mathematics instruction that create a rich environment for students to develop their mathematical thinking. As such, it is hoped that a synthesis of math instruction and student learning, as well as incorporating mandates from curricular frameworks, will enable myself and other teachers to recognize emerging principles for mathematics instruction that correlate with improved student understanding.

As each learner and classroom is unique, it is not possible to prescribe a specific list of best practices in mathematics instruction. However, if common elements can be identified across research, they can then be adapted appropriately by a teacher professional as instructional frameworks in his or her classroom. It is my desire to demonstrate how these common elements in mathematics research can be applied to several curricular outcomes in order for myself and other teachers to see the potential benefits in adapting these principles and practices for the classroom.

Question

What are the common themes between mathematics education research in North America and curricular frameworks of the National Council of Teachers of Mathematics (NCTM) and the Western and Northern Canadian Protocol (WNCP)? Following this, how can these common themes be applied to the teaching and learning of mathematics in secondary classrooms?

In order to answer this question, we will begin with an examination of research regarding the teaching and learning of mathematics in the context of North America. We will then analyze the frameworks supported by the NCTM and the WNCP. From these three groups, common elements concerning mathematics instruction and learning will be determined, cross-examined and condensed. Once these common themes are identified they will be applied to several curricular outcomes in the WNCP Grades 10-12 Common Curricular Framework as illustrative examples.

Literature Review

Math Education in North America

In conducting the literature review, thirty-two academic articles were compared from ProQuest, ERIC, and JSTOR databases. The review was conducted between October 2009 and March 2010. Common search terms for the review were “mathematics education,” “research studies,” “trends,” and “issues.” In reading multiple articles from a variety of academic journals, it was surprising to find recurrent themes. The following discussion relates to these recurrent themes.

The most common theme in the articles reviewed was the need to place less emphasis on procedural math (Baroody and Hume, 1991; Cavanagh, 2008; Cobb and Yackel, 1996; Dreyfus and Eisenberg, 1986; Garnier et al, 2005; Lampert, Penelope, and Putnam, 1990; Myers, 2007; Sinclair, 2009). None of the other articles reviewed suggested an increase of importance on teaching and following algorithms in class. Yet this is the typical emphasis in the North American mathematics classroom.

Garnier et al (2005) describe the all too common classroom in North America. The teacher begins the class by asking the students if they have any questions from the homework in the previous class, requiring the teacher to write solutions to a few questions on the board. The teacher then proceeds with the new lesson for the day. The topic is introduced as notes on the board. The teacher shows how to solve computational questions related to the topic and often provides a step-by-step procedure for the students to follow. Following the teacher’s examples, the students are given questions to solve from the board while the teacher walks around and checks their work. The procedure to solve the questions is placed on the board and the ensuing questions from the textbook are assigned. The teacher sits down and students are instructed to work on the assignment independently in their own desks. This scenario is significantly different than the principles offered by the researchers in the following section.

Mathematics education in North America does not “add up.” Garnier, Gallimore, Givvin, Hollingsowrth, Hiebert, Jacobs, Manaster, et al. (2005) summarize U.S. eighth-grade mathematics teaching as “frequent review of relatively unchallenging, procedurally oriented mathematics during lessons that are unnecessarily fragmented” (p. 125). In an interesting comparison, nations that scored higher than the U.S. in the Third International

Mathematics and Science Study had a greater percentage of questions as application problems (Garnier et al, 2005). Japan, for example, averaged 74% of all problems as application questions compared to 34% in the United States (Garnier et al, 2005). The benefit of this is that application problems place the question in context, allowing students to construct meaning.

Hand, Nasir, and Taylor (2008) conducted a study among students that were not successful in solving problems in a mathematics classroom. They found that when given an applicable context of interest, such as solving basketball average questions, students were able to utilize alternative strategies to solve the questions. The methods they used were not taught in class, but they relied on intuitive understanding and exploration to solve the questions. Hand, Nasir, and Taylor (2008) acknowledge that success of contextual math over rote memorization and procedural math has long been known. They quote the National Education Association in 1899 as describing the benefits of allowing students to construct their own models rather than by presented with prefabricated questions. It is interesting to note that a philosophy of teaching in mathematics education proclaimed over one hundred years ago is still struggling to take effect. Something needs to be done before the same mistake is made again for the next one hundred years.

Placing questions in context also helps students correlate problem solving in the real world says Schoenfeld (1989). There is a fundamental difference between problem solving in a typical North American classroom and the real world. One approach takes more time. Schoenfeld (1989) laments that North American educators have created a culture of students, through common homework problems and test problems that give a concrete result in one or two minutes, that are not willing to persevere in problem solving. Any problem taking more than twelve minutes, Schoenfeld states, will have most students believing that it is impossible to solve. As many adults can attest, most problems in the real world can hardly be even understood in the first twelve minutes.

The implications of these findings suggest the importance of contextualizing problems. The temptation is to think that more questions or concepts is better, but covering more questions or teaching more concepts has not shown to correlate with students learning more mathematics (Romberg and Carpenter, 1986 as cited in Confrey, 1990).

Cauley and Seyfarth (1995) also advocate for students learning mathematics in context. They argue for "engaging problematic situations" that can be approached from a variety of ways. Real problems in life do not present themselves as strictly algebraic or as only to be solved by applications of rates. Problems presented in the classroom should be similar in nature to problems encountered in the real world. In solving real-world problems, guidance and suggestions may be given, but only as necessary to keep the student engaged. Allowing students to explore problems in context from multiple approaches also places less emphasis on procedure.

Baroody and Hume (1991) and Meyer (1997), suggest several principles for effective math instruction. Meyer (1997) recommends that instruction should start with experientially real activities from which students can abstract concepts, create and model their understanding, interact with others through reasoning, and be intertwined with other strands rather than being independent. Baroody and Hume (1991) suggest a focus on understanding over rote memorization, encouraging active, purposeful learning, fostering informal knowledge, linking instruction to informal knowledge, and encouraging reflection and discussion. These suggestions have three common elements: building on background knowledge, reasoning, and communicating through discussion.

First, Baroody and Hume (1991) and Meyer (1997) emphasize the importance of drawing on student's background knowledge. Although a student's mathematical intuition may be hidden by years of procedural emphasis (Sinclair, 2009), teachers must allow students to explore rich problems using existing knowledge. Folk and Van de Walle (2004) also argue that knowledge is built upon a learner's construction.

Second, this background knowledge can then be built upon as students infer and conjecture possible patterns and solutions through reasoning. Rather than suppressing intuition and exploration by detailing solutions on the board before students are able to engage in a problem; math instruction must build on what students already know (Dreyfus and Eisenberg, 1986) in leading to more abstract and complex concepts.

Third, reflection and discussion is essential. In addition to Meyer (1997) and Baroody and Hume (1991), Anthony and Walshaw (2008) also stress the importance of dialogue and communication among students in math instruction. Teachers should not assume that this happens naturally between students. Students need to be guided in thinking together, asking challenging questions, and checking the plausibility of differing solutions through reasoning. Cobb and Yackel (1996) extend this idea further by insisting that guided instruction needs to lead students to appreciate the beauty of math, or the aesthetic, through understanding difference, efficiency, elegance, and connections.

Effective mathematics instruction is not limited to connections of contextual problems within mathematics. Mathematics can and should be applied to other disciplines such as English (Seo, 2009), art (Grzegorczyk and Stylianou, 2005), music (Rogers, 2004), and science (Mumford, 2006). An example from each discipline will be given to help illustrate the point.

Seo (2009) argues that connecting math and English can be beneficial for grade nine and ten students who have difficulties understanding grammatical structure in the writing process. By using the concept of an equation, Seo encourages students to form paragraphs similar to the way that they would solve an equation. One begins with a topic sentence and each subsequent sentence relates back to the initial sentence. Seeing paragraph construction in pictures and symbols, Seo (2009) argues, can help students improve their writing.

Art has many connections to mathematics, particularly in the area of geometry. Dutch artist, M. C. Escher created numerous drawings that show tessellations, reflections, dilations, translations, and rotations – all components of transformation geometry. In the same context, Grzegorczyk and Stylianou (2005), suggest connecting geometry and art through integration of symmetry.

Music is mathematical. Rogers (2004) shows how musical acoustics can be integrated in mathematics and science. Octaves, scales, semi-tones, frequencies, measures, and acoustics can be expressed numerically. These values can then be added, subtracted, multiplied or divided to create similar, yet unique forms of music.

Mary Somerville, a prominent female mathematician in the nineteenth century, is quoted as saying that "All of math and science is interconnected" (Reimer and Reimer, 2005). Mumford (2006) would agree. He advocates for math, integrated into the sciences, to be hailed as a relevant way to understand the physical world. This cannot be done without connecting math to science. Several of the examples he includes are the measurement of motion, state variables of a wave, cell biology, the significance of graphing, and the necessity for average citizens to understand the p-value in statistical analysis.

English, art, music, and science are only a few examples of the possible connections of math to other subject areas. In the teaching and learning of mathematics in the secondary classroom, connecting math to other disciplines should be a planned priority.

In summary, the current emphasis in North American math education on procedures (Baroody and Hume, 1991; Cavanagh, 2008; Cobb and Yackel, 1996; Dreyfus and Eisenberg, 1986; Garnier et al, 2005; Lampert, Penelope, and Putnam, 1990; Myers, 2007; Sinclair, 2009) needs to be replaced with students solving problems (Schoenfield, 1989) in context (Cauley and Seyfarth, 1995; Hand, Nasir and Taylor, 2008) by constructing knowledge (Folk & Van de Walle, 2004; Meyer, 1997) through reasoning (Barody & Hume, 1991; Meyer, 1997), communicating (Anthony and Walshaw, 2008; Baroody and Hume, 1991) and making connections to other subject areas and the world (Mumford, 2006).

National Council of Teachers of Mathematics

The National Council of Teachers of Mathematics (NCTM) was founded in 1920 in the United States and still exists today as a public voice in supporting teachers to provide the highest quality mathematics education for all students (NCTM, 2010). Since 1980, the NCTM has called for radical changes in mathematics education (Higgins, 1997). Publications from the NCTM have outlined a shift from procedural computations to developing students' mathematical thinking (Graham & Fennell, 2001). For example, *Principles and Standards* (2000) states that the main focus of math instruction is student understanding. This is then demonstrated through the themes of deemphasizing procedural math, allowing students to construct their own knowledge, encouraging students to infer and conjecture through reasoning, encouraging communication through reflective thought and discussion, and providing learning situations in which students are able to make connections.

In 1989 when *Curriculum and Evaluation* was published, a precursor to *Principles and Standards*, the NCTM received criticism for directing math instruction away from computational knowledge to deeper understanding through problem solving (NCTM, 2000). Despite the criticism, the NCTM still maintains this focus stating in 2000 that learning with understanding should be the goal of mathematical instruction. Learning from understanding comes from students "actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 11). As autonomous learners, students construct meaning when given appropriate tasks in context. When students do not rely on the teacher as the transmitter of knowledge but rather become independent learners their depth and breadth of learning is extended (NCTM, 2000). Constructing knowledge then is closely connected with learning with understanding.

In learning with understanding, it is not enough for students to construct their own mathematical reality. Students need to infer, justify claims, and prove conjectures (NCTM, 2000) to test their constructs. Reflective questions such as "Does this work in situations other than this context? Can I think of one example that makes this statement false? Is there a pattern developing that I have seen before?" are questions that test a students' ideas. Rather than asking a teacher to judge the validity of an idea, students must become proficient in testing their own ideas with their previous knowledge. They also can benefit from reflecting and discussing with their peers through communication.

Communication is listed as one of the standards in *Principles and Standards*. Four goals of instructional programming related to communication are expressed in this standard. First, communication is important in order for students to be able to conceptualize their own mathematical thinking. They need to be able to sort and frame their ideas in order to be able to share their thinking. Second, in order for the listener to make sense of their ideas, students must be able to communicate effectively. Effective communication includes explaining concepts in a clear and logical manner. Third, communication affords students the opportunity to critique one another's thinking. Multiple viewpoints can help a student see the validity or inconsistency of a construct. Fourth, communication forces students to develop mathematical terminology. In order to express concepts accurately, mathematical language needs to be utilized and mutually understood. Therefore, communication is crucial to developing student understanding in mathematics.

Another important aspect of developing student understanding in mathematics is connections. In *Principles and Standards* (2000) three domains of connections are outlined. The first domain encourages students to recognize and use connections within mathematics. Students can prompt themselves by repeatedly asking themselves how the current concept relates to concepts they have examined before. Relating geometry to algebra or relating ratio to fractions and decimals are examples within this domain. In the second domain students need to consider how individual concepts combine to create a coherent whole. Students must build upon what they already know in order to connect concepts. For example, a student could determine the surface area of a cylindrical juice can by examining the individual faces. Although a student may not have calculated the surface area of a cylinder before, the student could use prior knowledge to add together the areas of two circles and the rectangle. The third domain applies to contexts outside of mathematics. Mathematics is not an isolated subject of study. As was illustrated earlier, rich connections can be made to other subject areas such as English, art, music, and science.

In reviewing the directives from the NCTM in order to develop students' understanding, five themes emerge. First, less emphasis should be placed on following procedures and more emphasis placed on problem solving. Second, students need to build upon prior knowledge and experiences in constructing meaning in mathematics. Third, opportunities to infer, justify, and prove should be provided to students. Fourth, communication in organizing concepts, sharing effectively, evaluating other's ideas, and developing mathematical language is necessary. Last, making connections within mathematics and outside of mathematics is important for providing a context.

Western and Northern Canadian Protocol

The Western and Northern Canadian Protocol (WNCP) is a curricular framework that is shared by the four western provinces and three northern territories in Canada (WNCP, 2008). It began as the Western Canadian Protocol in 1993 to encourage collaboration in primary and secondary education. Within mathematics, this collaboration was not fully actualized until 2008 when, supported by the ministries of education in each jurisdiction, the *Common Curriculum Framework for Grades 10-12 Mathematics:*

Western and Northern Canadian Protocol was published (WNCP, 2008). In the introduction and conceptual framework of the publication, the purpose, beliefs, goals, and mathematical processes of the WNCP are outlined. It is within this content that principles for the teaching and learning of mathematics will be examined.

Similar to the mandate of the NCTM, the WNCP states, "learning through problem solving should be the focus of mathematics at all grade levels" (2008, p. 8). In describing authentic problem solving, not all types of questions are considered to be equally valid. For instance, when students are given ways in which to solve the problem it is considered practice, not problem solving (WNCP, 2008). The emphasis is on students accessing prior knowledge and using it in new contexts rather than following procedures. Suppose students were shown how to solve trigonometric ratios in right-angle triangles. If a word problem were then presented to students in which they had to solve for a missing side length or angle, this would not be considered problem solving, but rather practice. Instead, an example of a problem that allows for multiple methods of solving could ask students to calculate the height of a tree situated on the school grounds. Students could work in small groups and use a variety of methods such as similar triangles, trigonometric ratios or ratios of shadows in order to calculate the height of the tree. In this situation, students would need to build upon their previous knowledge in order to problem solve.

Related to problem solving, building upon previous knowledge, expectations, and experiences is important in developing mathematical understanding (WNCP, 2008). The WNCP provides three guidelines for students constructing meaning. First, mathematical experiences should proceed from simple to complex and concrete to abstract. Second, a wide variety of manipulatives, tools, visuals, contexts, and instructional styles should be available to students as each individual student tries to construct meaning of a new concept. Third, discussion between students is necessary for students to build connections between concrete and symbolic representations of mathematics. Relevant discussion enables students to test their newly formed ideas with others.

Testing ideas through inference, analysis, and justification allows students to develop mathematical reasoning (WNCP, 2008). Students should be continually challenged to question why they believe something is true. Responsibility is taken from the teacher and placed on the student in deciding if a conjecture is valid. This also fosters an environment in which students can become autonomous learners, connecting mathematical reasoning to contexts within and outside of mathematics.

Making connections is related to the previous three concepts of problem solving, constructing knowledge, and reasoning. As much as possible, problems should be connected to meaningful contexts (WNCP, 2008). For "through connections, students begin to view mathematics as useful and relevant" (WNCP, 2008, p.7). Reasoning is also linked to connections in that students need to relate previous concepts that they have learned in mathematics to the new ones that they are constructing. Making connections helps students make sense of mathematics as a whole and helps them see ways in which mathematics is integrated into other subject areas and everyday life (WNCP, 2008).

Communication is a critical aspect of learning, doing and understanding mathematics (WNCP, 2008). In order to express ideas, attitudes, and beliefs about mathematics, communication through language is necessary. Communication promotes the development of mathematical language through students making connections between simple and complex, concrete and abstract mathematical ideas (WNCP, 2008).

In summary, there are five major principles and practices that come forward from the WNCP Common Curricular Framework. The five themes are focusing on problem solving, providing opportunities for students to construct knowledge, challenging students to reason, encouraging students to make connections, and fostering communication between students.

Synthesis

The first part of my research question asked about the common themes in mathematics education research and curricular frameworks of the NCTM and WNCP. These have been identified in each of the three areas and will be cross-examined. Following this cross-examination, the second question, how these themes can be applied in the secondary classroom, will be outlined using curricular outcomes.

Five themes emerge from the NCTM, WNCP, and mathematics education research. First, there should be less of an emphasis on procedural math and more of an emphasis on integrated problem solving. As stated earlier, this was the most often reoccurring theme in the research. Learning through problem solving is also described by the NCTM as "an integral part of all mathematics learning" (NCTM, 2000, p. 51) and by the WNCP as the "focus of mathematics at all grade levels" (WNCP, 2008, p.8).

Second, building upon students' prior knowledge and experiences in order for students to construct new learnings was emphasized in the research (Barody and Hume, 1991; Folk & Van de Walle, 2004; Meyer, 1997) and mentioned as a predictor of success in the WNCP (2008, p. 2) and as "essential" in the NCTM (2000, p.10). Constructing knowledge is the basis for creating meaning to mathematical concepts.

Third and fourth, reasoning through justifying and proving was also stressed by each of the three groups as was the fourth common theme of communication. Building on inferences to create deeper meaning (Dreufus and Eisenberg, 1986) was identified in the research and paralleled as "essential to understanding" (NCTM, 2000, p. 55) and necessary in order to make sense of mathematics (WNCP, 2005). Communication brings meaning to ideas (NCTM, 2000) through a variety of forms to create connections (WNCP, 2008). Anthony and Walshaw (2008), Baroody and Hume (1991), and Meyer (1997) also emphasized communication as essential in developing students' mathematical thinking.

Fifth, the reviewed research articles, the NCTM, and WNCP accentuated the necessity of making connections both within mathematics and outside of mathematics. Connections were seen as "powerful in developing understanding" (WNCP, 2000, p. 7) and capable of producing "deeper and longer lasting understanding" (NCTM, 2000, p. 63). Examples of deeper understanding through connections to other subject areas and within mathematics were also demonstrated (Mumford, 2006; Rogers, 2004).

In summary, in order to develop students' mathematical thinking and understanding, students should learn through problem solving by building on previous knowledge and experiences, justifying the reasonableness of their constructions through discussion and make connections to contexts within and outside of mathematics.

Applications

The five common themes identified earlier will be applied to three topics within the WNCP Grades 10-12 Common Curricular Framework. I chose the WNCP framework because I am a teacher situated in western Canada. The actual examples chosen are only important to illustrate the applications of the common themes. Each teacher needs to study his or her jurisdiction's curricular mandates as well as the context of the learners to determine appropriate applications. The three topics I have chosen are magic squares, sine and cosine law, and measurement. These topics are a part of the WNCP framework and are listed by their corresponding curricular outcomes in the proceeding section.

Magic Squares

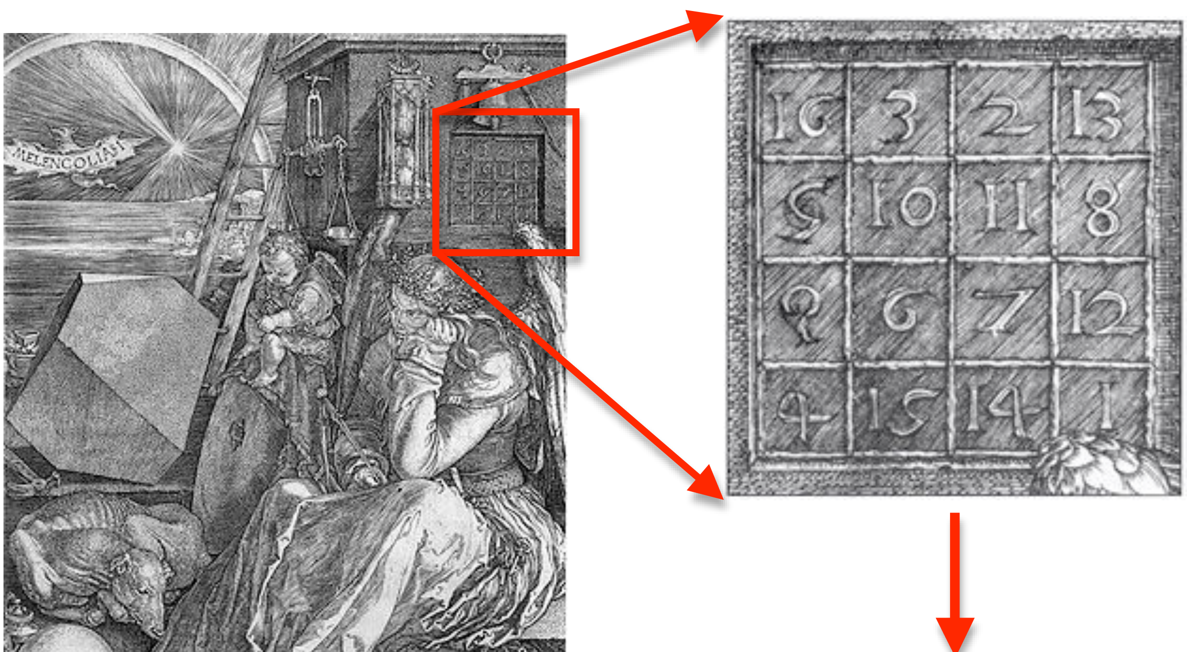
Magic squares can be found in the WNCP Curricular Framework for Grades 10-12 in the Foundations 30 pathway. Foundations 30 is a grade twelve course designed for students entering arts and sciences in post-secondary education. Under the general outcome of logical reasoning, specific outcome number one states:

> It is expected that students will analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. *(It is intended that this outcome be integrated throughout the course by using games and puzzles such as chess, Sudoku, Nim, logic puzzles, magic squares, Kakuro and cribbage.)* (WNCP, 2008, p. 69).

To begin, there is nothing "magic" about magic squares. This topic can be taught in a traditional manner with students following procedures or it can be taught in ways reflecting the previous research examined and the NCTM and WNCP frameworks. The topic is not important as compared to the way in which it is taught. In Appendix A, I have included a student activity guide with questions and solutions on magic squares. As I discuss this topic, it may be beneficial to reference the complete lesson plan in Appendix A.

We begin with a problem situated in a context. In 1514 Albrecht Dürer created a wood engraving entitled "Melecolia." In the upper right-hand corner of the artwork there is an arrangement of numbers in a square with 4 rows and 4 columns.

"Albrecht Dürer's Melecolia"



(Credit: Metropolitan Museum of Art, 2010)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A teacher could begin by asking students in groups to write out their observations about the numbers. A set number of observations is not mandated, students are simply asked to make observations about the arrangement of numbers. Most students are probably familiar with magic squares and may notice that the sum of each row, column, and diagonal is 34. In groups, students should be given time to explore further properties of the square. After an appropriate amount of time is allowed, groups of students should be encouraged to share with one another concerning observations they have made. It is quite likely that some groups will notice things that others did not. Further, in order to communicate, students will have had to use terminology to express their findings. For example, students may have created a phrase to describe the constant of the magic square. They may not have used that exact word but in making observations and explaining them to other students they would have communicated the idea. Communication can occur by providing an opportunity for students to share their observations about mathematics.

If students are having difficulty making observations beyond the constant of the square, the teacher can help the students. However, the teacher does not help the students by giving them the answer, rather the teacher can act as a facilitator in prompting the students. In this example, a teacher may ask, “Do you notice this sum anywhere else in the square?” If a student is not equipped to ask reflective questions of him or herself a teacher can help develop that type of mathematical reflection by asking the student.

The teacher could also ask, “How do you think Albrecht Dürer made this arrangement?” Students may notice the resemblance of Dürer’s square to a square in which the numbers from 1 through 16 are written in sequential order. Building on what they already know about transformations, students may be able to determine which numbers have been translated to create the magic square.

“4 by 4 square in sequential order translated to Dürer’s square”

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Another possible prompting question for students could be “Do you see anywhere in the square where Albrecht Dürer may have left his signature?” There are several implications of this type of discussion. First, students may see that not everything in mathematics is certain. There is no way to know with certainty that the 15 and 14 refers to the year and the numbers 4 and 1 refer to his initials “D” and “A” based on their position in the alphabet. Second, it helps students make connections outside of mathematics. Students may notice the umlaut in the artist’s last name and question his country of origin. If the class ended at this point, I speculate that some students would continue investigating this question. A discussion may arise questioning if the German language has the same alphabet as the English language. If it did not, the conjecture that the 4 and 1 refer to his initials would most likely not be true. If however, the letters of the alphabet and the order is the same as the English alphabet, at least for the first four letters, it is likely that the conjecture is true. This question allows students to build and connect knowledge about foreign language to mathematics.

With a rich task, even a simple observation can be extended to develop mathematical thinking through reasoning. As mentioned earlier concerning Dürer's 4 by 4 square, most students will notice that each row, column, and diagonal has a sum of 34. The question "why" can often challenge students to think about the mathematics behind a concept rather than merely accepting it because that is what they were told. Constructing knowledge is rooted in students making sense of mathematical concepts (Folk & Van de Walle, 2004) instead of memorizing and practicing procedures. To extend the observation of the constant as 34, a teacher could ask, "Why is 34 the constant?" A quick response may follow based on the sum of each row, column, and diagonal. To this a teacher could respond, "What is the constant of a 3 by 3 magic square?" One student may try and generate a 3 by 3 square in order to answer the question and another student may recognize that the sum of the numbers in a magic square divided by the size of the square is the constant. There are multiple methods and reasons for students to come to this conclusion. These methods can be shared and tested among classmates. Their conjecture can be tested further by asking them, "What is the constant of a 5 by 5 magic square? A 6 by 6 magic square? A 7 by 7 magic square? A N by N magic square?" The goal is to move from the concrete to the abstract. By starting with something that is tangible and can be seen, such as 3 by 3 or 4 by 4 magic square, students can then build on what they know and construct meaning for an abstract concept such as the constant of a N by N magic square.

Students could also be asked in groups to generate all possible 3 by 3 magic squares. Once again, students would have to use language in order to describe the different possible magic squares. They would also have to reason in showing that there were a certain number of solutions and no more. It is likely that students will disagree as to the number of solutions, creating the possibility for engagement and dialogue. In particular, this question is excellent in helping students see the need to be able to clearly explain and present their solutions. As there are 7 rotations and reflections of the one unique magic square, students are likely to be challenged in representing all of them. The students in the class should decide what counts as a unique solution. Discussion may be connected to rotations and reflections or it may be related to an algebraic or visual representation.

There are still more ways for students to develop their mathematical thinking through magic squares. Students could be given a problem that they are not able to solve. A teacher could start by asking students in groups to create a 7 by 7 magic square. Most students would attempt this task for several minutes and then become frustrated by the level of difficulty required of them. However, it is important to note that prior to this question students have not been given examples and step-by-step procedures in creating a 7 by 7 magic square. They will need to build on their understanding of previously learned concepts such as average, weighted mean, ratio, and patterning in order to solve the question.

Teachers must help facilitate student learning. If a teacher gave the answer directly to the students and then had the students replicate the process, very little learning would occur. Instead the role of the teacher is to help students develop mathematical thinking. Using the previous example, a teacher could help students by questioning them. A teacher might ask, “Have you seen a problem like this before? Can you break it down to a simpler problem?” Some groups of students might look at the 3 by 3 magic square and look for patterns in it. Others might try and use the constant value to determine columns and rows. After a while, it is likely that most students would not be progressing in being able to generate a 7 by 7 magic square. At this point, the teacher could give each group of students a completed 5 by 5 magic square with the number 1 in the middle position of the top row.

"Completed 3 by 3 and 5 by 5 magic square with 1 in the top middle position"

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Students could be encouraged to look for patterns between their completed 3 by 3 square and the 5 by 5 square. They would need to create a conjecture and then test it. Most students will not think that they are capable of ever creating a 7 by 7 magic square. However, by giving them an opportunity to recognize a pattern and apply their understanding of constants, it is possible that many students will be able to generate a 7 by 7 magic square. When students arrive at a correct solution when they did not think they could ever solve question, it helps build confidence. If students are successful in determining the pattern from the magic squares above, they can be challenged to discover other generating patterns. They can then reflect, discuss, and justify the situations in which their pattern works. Rich problems allow students of all levels to access the question and also provide stimulating extensions for others.

One other such extension is the creation of a 9 by 9 magic square using a Sudoku grid. Instead of thinking of the square as only 9 rows by 9 columns, students can be challenged to create a magic square in which each 3 by 3 square is a magic square and the sum of any three 3 by 3 squares in a row, column, or diagonal is also equal.

"Completed 9 by 9 square with Sudoku grid"

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

A complex question such as this requires students to build upon their previous knowledge and apply it in a new situation. If students had been taught a procedure in creating standard magic squares, it is likely that many would find it difficult to complete this task as the definition of a magic square has been altered. An important discussion during this task and after is for students to consider why this arrangement works. It is probable that students will make natural connections from the arrangement of this magic square to Sudoku. Making connections, as argued earlier is an important aspect of learning mathematics.

Choosing appropriate tasks to meet curricular outcomes is critical. Tasks need to be problem based and allow students to construct understanding through reasoning, communicating, and connecting. In *Principles and Standards* (2000), the NCTM describes four benefits of choosing appropriate tasks. First, students gain confidence by tackling difficult problems (NCTM, 2000). At the beginning of the lesson, most students would not think that they are capable of generating a 7 by 7 magic square. However, by helping students break down the question into a simpler question and applying what they know, it is possible for students to be successful at the task. There is a sense of satisfaction, possibly an appeal to the aesthetic, when a student is near completing a large magic square and he or she begins checking the solution by adding the rows, columns, and diagonals. This sense of satisfaction is not as great if the student had been told what to do and had followed the instructions.

Second, students are eager to find things out on their own when presented with an appropriate task (NCTM, 2000). When wonder is invoked in students, they become intrinsically motivated. They are not concerned whether this information is going to be on a test or when they will use this in real-life, they are instead engaged in the learning and desiring to learn for themselves. From Albrecht Dürer's engraving, some students may wonder if his initials correspond to the numbers in the bottom row. Others may be curious if it possible to generate a 6 by 6 or 8 by 8 magic square by using the translations in Dürer's 4 by 4 magic square. Still others may wonder if it is possible to prove the number of all possible 4 by 4 magic squares.

Third, appropriate tasks afford students flexibility in exploring mathematical ideas and trying alternative solution paths (NCTM, 2000). Students are not limited to following a set procedure multiple times. Instead, rich mathematical tasks allow students to think critically about how to solve a problem. Alternative solutions paths can also lead to student led discussions. In this environment, the students, rather than the teacher, direct the discussion and refute or demonstrate the validity of possible alternative solutions.

Fourth and last, when appropriate tasks are chosen for students, they are willing to persevere in completing them (NCTM, 2000). Perseverance develops from intrinsic motivation and confidence and is necessary to solve most problems. However, this is not the message most students receive from homework assignments. As discussed before, Schoenfeld (1989) found that because of categorized practice questions that condition students to quick answers, most students believe that any problem that takes longer than twelve minutes to solve is impossible. Appropriate tasks demonstrate to students that solving problems takes time and perseverance is necessary in order to solve them, similar to solving problems in the real world.

The five identified common themes have been applied to the teaching and learning of a curricular outcome, namely developing logical reasoning through magic squares. It is hoped that through this example, other teachers will be able to apply the concepts to other curricular outcomes. In order to further see possibilities and benefits of teaching from this philosophy, two more examples will be given.

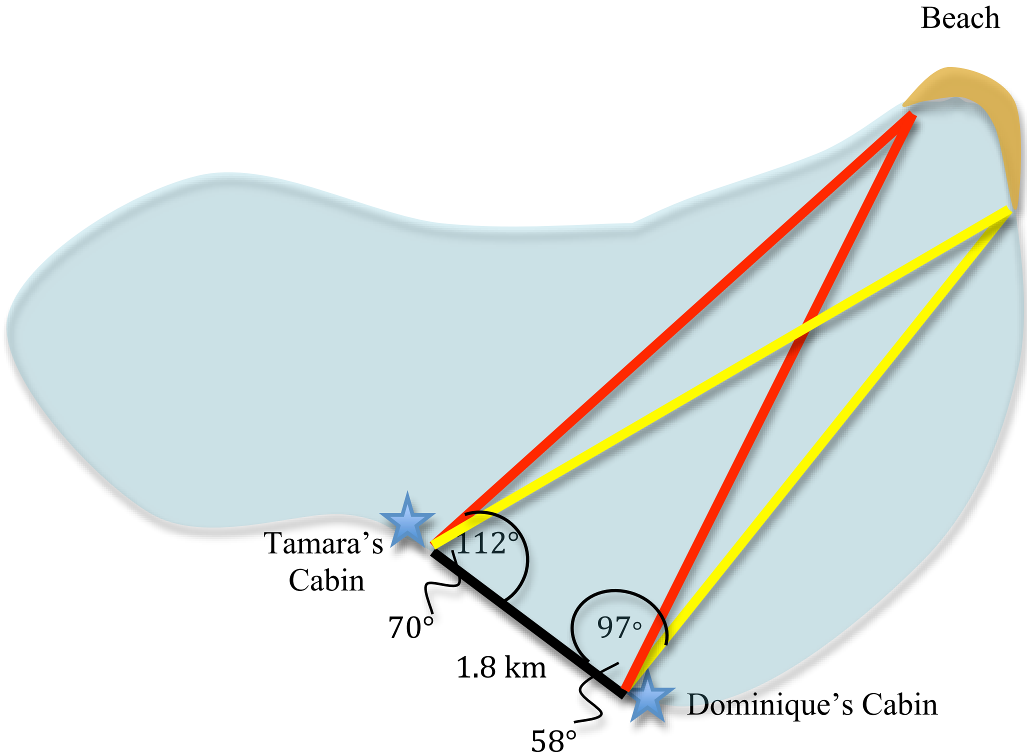
Sine and Cosine Law

Solving problems using the sine and cosine law is an outcome that is found in all three of the pathways described in the WNCP. In Foundations 20, a grade eleven course, outcome three in geometry expects students will "solve problems that involve the cosine law and the sine law, including the ambiguous case." (WNCP, 2008, p. 59). Also, under trigonometry in Pre-calculus 20, another grade eleven course, outcome number three states that "it is expected students will solve problems that involve the cosine law and the sine law, including the ambiguous case." (WNCP, 2008, p. 81) Further, in Apprentice and Workplace Mathematics 30, a grade twelve course, outcome number one in geometry expects students will "solve problems by using the sine law and cosine law, excluding the ambiguous case." (WNCP, 2008, p. 40).

In Foundations 20 and Pre-calculus 20 the achievement indicators expect that students not only solve problems using the sine and cosine law, but that they are also able to generate the sine law and cosine law (WNCP, 2008). Accordingly, a problem is presented at the beginning of the lesson that students will not be able to solve using trigonometric ratios in a right-angle triangle. The lesson enables students to derive the law of sines based on their previous knowledge of trigonometric ratios. The students then use this law to solve the original problem. A similar lesson framework with a different problem is also presented for the law of cosines.

To begin the lesson a problem is presented in context. Two friends are contemplating canoeing from one of their cabins to a beach across the lake. They want to know the shortest distance to canoe. For a full description of the question and lesson plan see Appendix B.

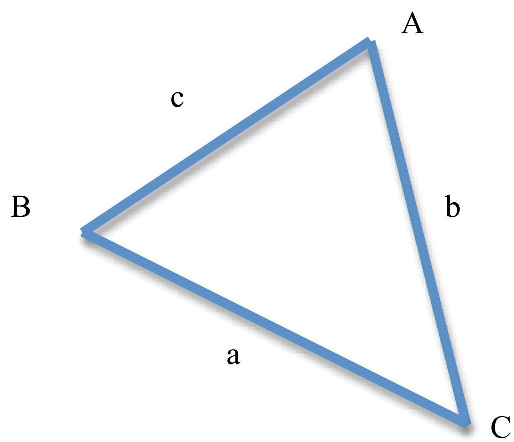
"Diagram of Canoeing Question"



The question can be considered a problem, as students have not already been given ways to solve it (WNCP, 2008). Students may recognize the similarities between this problem and solving trigonometric ratios. However, most students will not know how to use trigonometric ratios in triangles without a 90° angle. This can lead to an interesting discussion between students if it is possible to still use sine, cosine, and tangent in non-right angled triangles. As the teacher, it is important to encourage this type of conversation rather than give a direct answer or move too quickly to the next concept. A teacher can validate students' mathematical thinking by providing time for students to question and wonder.

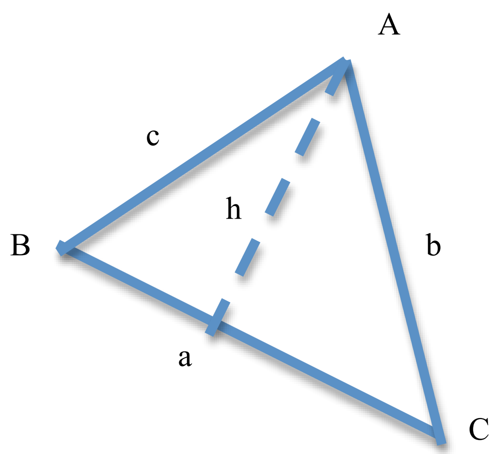
In discussing the use of trigonometric ratios in oblique (non-right angle) triangles students may have tried to test their thoughts by using examples and counter-examples. Often students begin at the concrete level, using specific measurements for side lengths and angles. At this point, a teacher could ask students, "How is it possible to show a relationship for a general oblique triangle?" Prior to grade eleven math, it is likely that students have had experience in proving congruency for general triangles. The suggestion may be made to draw a general oblique triangle. An example is shown below.

"General Oblique Triangle ABC"



Different groups of students will draw different sizes of oblique triangles and label the sides with different letters. This will only help students see that this relationship works for all oblique triangles and not just special cases. The teacher can ask guiding questions of the students as necessary. Starting at the concrete the teacher can ask students to consider everything they know about triangles and finding missing lengths and angles. Students should know certain concepts such as given two angles in a triangle the third can be calculated, if one side length and one angle is known in a right-angle triangle the other missing lengths and angles can be found, and given the length of two sides of a right-angle triangle the third side can be calculated. Students need to connect the present problem to their previous understanding. One group of students may recognize that a general oblique triangle can be divided into two smaller right-angle triangles by drawing the altitude from one of the bases.

"General Oblique Triangle ABC with altitude at A"



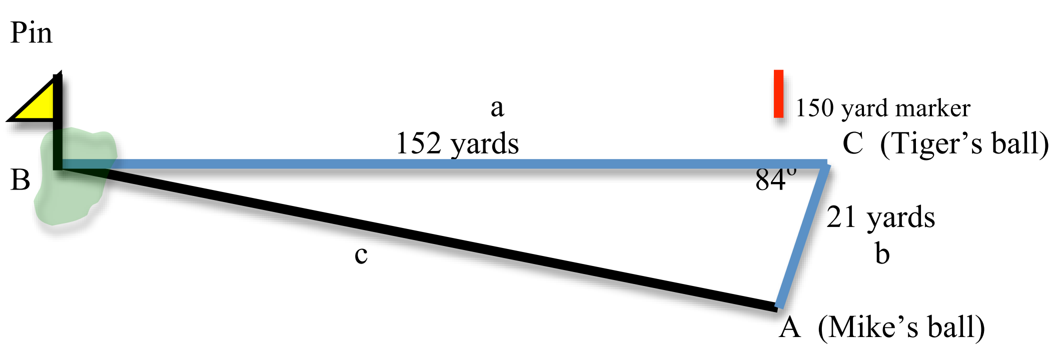
Most students will recognize the potential to use sine, cosine, and tangent in the two smaller triangles but will not know which relationships to use. Without directly stating the relationship the teacher can ask the students guiding questions such as, "If we want to relate the altitude to two different angles, which trigonometric ratios should be used?" By continuing this process in a similar manner, students can determine the law of sines. The full list of guiding questions is outlined in the lesson plan in Appendix B. Students can then use this relationship to solve the original problem.

A traditional method of showing students a relationship, giving some examples and then having the students practice individually does little to develop students' mathematical thinking (Garnier et al, 2005). Students need to construct their own meaning from connecting relationships. With the law of sines, students' understanding would not be greatly increased if the teacher introduced the topic, gave some examples and then the students practiced using it on triangles. Problems that students are not immediately able to solve can help students see the need to learn new concepts.

Once students have derived the law of sines, several extensions and connections can be made. In solving the canoeing problem, students will most likely have used the sine law to relate two known angles and one known side in order to find the missing side length. Students may either wonder or can be asked, “Given various side lengths and angles of a triangle, when is it possible to use the law of sines to determine the missing lengths and/or angles?” Students will need to connect their previous learning of congruent triangles in order to determine the applicable cases. Work should be done in small groups so that students can test and verify each other’s solutions. In order to test their solutions, students could give two side lengths and one angle for a triangle that has zero, one or two solutions.

The law of cosines can also be derived in a similar manner. A problem in which students are not able to use trigonometric ratios of right-angle triangle of the law of sines is presented. It is best if the students have already derived the law of sines as the law of cosines requires additional steps and is slightly more difficult. The problem for the law of cosines is situated in golf in which two golfers are trying to determine which ball is closest to the pin after their initial drives.

“Tiger and Mike go Golfing”



Once again, the complete problem and sequence of questions in deriving the law of cosines is included in Appendix B. There are two extension questions of interest. First, students can be asked to figure out the applicable situations for using the law of cosines. Students may ponder or be asked by a teacher, "What is the minimum information in terms of side lengths and angles in order to use the law of cosines?" Second, there is an interesting case of the law of cosines when the included angle is 90°. For example, the golfing problem asks students to find the side length c if the angle at Tiger's ball was 90° instead of 84°. Students may notice that the value of Cos 90° is 0 and therefore the law of cosines is simplified to the Pythagorean theorem as seen below.

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$
$c^2 = 152^2 + 21^2 - 2(152)(21)(\cos 90^\circ)$
$c^2 = 23545 - 0$
$c = 153.44$ yards

The Pythagorean theorem is really a special case of the law of cosines. This is another relationship that can be seen from using problems that require constructing, reasoning, communicating, and connecting.

Folk and Van de Walle (2004) give seven benefits of using relations when developing understanding. First, it is intrinsically rewarding. Folk and Van de Walle (2004) contend that most people, especially children, enjoy learning. People enjoy making sense of new information and making connections. However, rote learning is not enjoyable for most as evidenced by numerous extrinsic motivators such as tests, rewards, or external personal pressure.

The second, third, and fourth benefits listed are closely connected. Relational learning enhances memory, it requires less for students to remember, and fourth, it helps with learning new concepts and procedures. Memory recall is not limited to isolated facts. If a student is not able to remember a concept, relational learning allows the student to reason through similar concepts with the possibility of connecting to the desired concept. In North American mathematics classrooms, reviewing of concepts accounts for almost one-half of the instructional time (Garnier et al, 2005). Instead of constant review of procedures and facts, relational learning allows students more time to study concepts in depth by relating them to concepts already learned, requiring less memorization for the student.

Improving problem solving is the fifth benefit of relational learning as described by Folk and Van de Walle (2004). When a student's learning is interconnected transferability between concepts is increased (Schoenfeld, 1992). Problems require students to access prior knowledge in order to form a solution. If a student is able to relate concepts within mathematics, rather than having each concept stored fragmentally, their ability to access combinations of this stored understanding is increased.

Sixth, relational learning is also beneficial because it is self-generative. Similar to improving problem-solving abilities, relational learning fosters the creation of new ideas and inventions. Students are more likely to demonstrate perseverance when confronted with a problem and also create new ways in which to solve problems.

Last, relational learning improves attitudes and beliefs. When students begin to feel that they understand mathematics and can solve problems, they develop a positive attitude toward mathematics. Mathematics is no longer seen as only accessible to the elite but rather understandable by everyday people. Recalling rote procedures may produce fear and anxiety (Folk & Van de Walle, 2004), but interconnected understanding is likely to produce a positive attitude toward the learning of mathematics as confidence correlates positively with achievement (Schoenfeld, 1989).

In review, relational learning is beneficial because it is intrinsically rewarding, it enhances memory, there is less to remember, it facilitates learning new concepts and procedures, it improves problem-solving abilities, it is self-generative, and it improves attitudes and beliefs. The law of sines and law of cosines were presented as an example as teaching in this way. One last topic, measurement, will be discussed in applying teaching and learning that is problem-solving based, allowing students to construct, reason, communicate, and connect their understandings.

Measurement

Linear, surface area, and volume measurement outcomes are included in two of the WNCP pathways. In the grade eleven course of Apprentice and Workplace Mathematics 20, under measurement, outcomes one and two state:

> Solve problems that involve SI and imperial units in surface area measurements and verify the solutions [and] solve problems that involve SI and imperial units in volume and capacity measurements. (WNCP, 2008, p. 30)

Also, in Foundations 10, three of the outcomes under measurement are applicable. It is expected that students will:

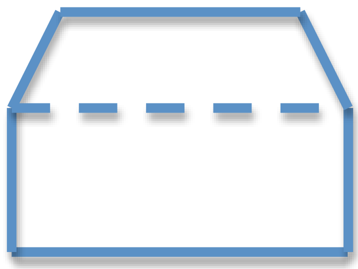
- Solve problems that involve linear measurement, using SI and imperial unit of measure, estimation strategies, [and] measurement strategies.
- Solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including right cones, right cylinders, right prisms, right pyramids, [and] spheres.
- Develop and apply the primary trigonometric ratios (sine, cosine,

tangent) to solve problems that involve right triangles.
(WNCP, 2008, pp. 47-48).

One possible way to meet these outcomes in a problem-solving context is to have students participate in a math trail activity. In the absence of categorized practice questions, students are forced to think critically about how to solve questions. As for the context, any school or building will have plenty of examples of objects or areas for students to measure. The following examples are taken from math trails that I have developed for the University of Saskatchewan and the University of Regina. For measuring tools, I give a similar length banana to each group of students because bananas do not have incremental marks and it requires students to convert between bananas in linear, squared, and cubed units. The five common themes will be applied to this activity by examining several of the tasks required of the students. For a complete description of this activity and questions, please refer to Appendix C.

Real-life situational problems help students construct meaning by building on previous knowledge. One of the questions asks students to calculate the volume of a glass elevator. The elevator is an irregular hexagonal prism. Students will most likely not have been taught the formula for finding the volume of such an object, but in standing in the elevator and attempting to measure its dimensions, they may notice that the hexagon can be divided into a rectangle and a trapezoid. They will have to build upon prior knowledge and experience to solve the problem.

“Diagram of elevator’s base”



Students should be able to determine the area of the rectangle and may be able to calculate the area of the trapezoid, recognizing that the area of the trapezoid is the same as a rectangle with a length that is equal to the average of the parallel sides. Students could also further compartmentalize the trapezoid as a rectangle and two triangles. One concept is easier to remember than several. Students who have previously made the connection that the volume of a prism is equal to the area of the base multiplied by the height instead of memorizing separate formulas for cylindrical, rectangular, triangular or pentagonal prisms are likely to realize that the volume of the elevator can be calculated if the area of the base is known. By adding together the areas of the trapezoid and rectangle and then multiplying by the height, the volume can be found.

Another task in which students need to build upon prior knowledge and connect concepts is in finding the volume of a regular hexagonal prism. The prism is surrounded with glass that extends to the ceiling so a simple measurement across the prism is not possible. In this context, the restrictions placed on the problem are not fictitiously imported but exist due to a physical obstacle. This allows students to see the need for creativity and multiple approaches in problem solving in real-life. Students may recognize that the hexagon can be divided into two equal trapezoids or six congruent equilateral triangles. By finding the area of one of the triangles, students could multiply by six to find the total area of the base. Seeing a regular hexagon as six congruent equilateral triangles may lead students to question if a rule for determining the area of a regular hexagon is possible if only the side length is known. If students substitute a variable such as “x” in every place that they used a specific number for the side length, they should be able to determine a formula for the area of a regular hexagon. Constructing new meaning by building on previous knowledge and experience can occur through real-life situational problems.

One of the main advantages of having students solve problems in real-life is the multiple ways in which students can solve the problems. Take for example, the question asking students to calculate the height of the flagpole in bananas. In order to solve the question, students could use proportions with shadow lengths, trigonometric ratios measuring the angle of elevation with a straw and a protractor, a bowl of water in which to see the top of the flagpole and similar triangles, scale by taking a picture with the banana and flagpole and then printing the picture and measuring it, or even throwing the banana vertically in the air to the height of the pole and timing its descent (this one gets a little messy). Students can become the experts in developing methods for determining the height of the flagpole. To be effective in sharing their idea, each group will need to communicate clearly and logically. As students become the judges of effective communication they can become more aware for co-constructing criteria in assessment for learning practices. In this way, students can take more responsibility in evaluating their own solutions, not relying solely on the teacher.

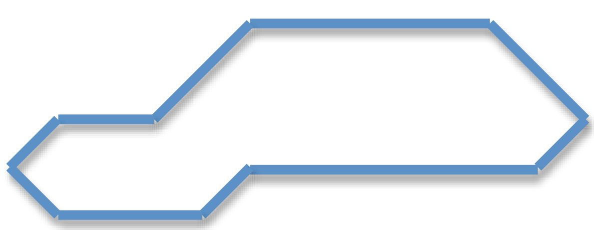
Communication is also necessary for students within each group. For example, part of one of the questions asks students to count the number of seats in a large auditorium.

As a group, students need to work together and communicate a systematic method of counting. Some aspect of recording and representing the groups of data is essential to ensure the precise number. Working together is not enforced in this situation, but students will quickly realize the need for everyone's participation and an effective means of organizing information and intercommunication.

Math trail activities can also help students identify connections outside of mathematics. One of the questions asks students to consider a tiling pattern on the floor and calculate the entire area of the geometric design. The geometric design forms an airplane with each portion of shaded tile either a sixteenth, eighth, quarter, half or whole portion of one tile. In summing the shaded portions of tiles, students will most likely observe the scale factor of the shaded tiles. They will be able to quantify the area mathematically, yet still be able to see the art of the design as a whole. The mathematics of the tiling contributes to the concept of the design.

Two other tasks also draw on connections between art and mathematics. In the geology building at the University of Sakatchewan, there is a planter that surrounds a replicated triceratops fossil. At first glance, the planter appears to be designed without patterning.

"Diagram of triceratops planter"

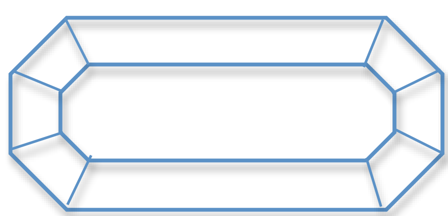


However, once students begin to measure the perimeter by measuring each side length, multiples of one number continually arises. Each side is a multiple of 5 bananas or 45 inches in length. Even though initially the polygon does not appear to have any order in its design, measuring the length of each side demonstrates that even apparent abstract art can have connections to mathematics.

A second task that connects to art is finding the area of multiple circles painted on a wall inside one of the university's buildings. As students are measuring the radii of various circles, they may notice that groups of circles are proportional to one another. Rather than repeating the process of measuring the radius twenty-three times, they may begin to infer which circles have equal radii. They may even notice how the sizes of the circles vary in relation to the arrangement and positions of the circles within the painting. What may appear as abstract art at first glance can reveal mathematic properties in visual aesthetics.

In addition to connecting to other subject areas, these tasks can also develop students' reasoning abilities. One task asks students to calculate the surface area of slanted faces around a monument. Students could calculate the surface area by measuring each of the eight trapezoidal faces but this is not necessary. Some students may notice that the monument is an octagon with vertical and horizontal symmetry.

"Diagram of Octagonal Monument"



Based on this observation they may wonder if they only need to find the surface area of four of the faces and then multiply the value by two. They may test their conjecture by measuring a fifth trapezoidal face and verifying that the measurements are congruent to its opposite corresponding face. By actually measuring distances students will most likely see the prevalence of mathematical concepts in everyday objects.

A second task in which students must reason is in estimating the flying distance between Vancouver, B.C. and Halifax, Nova Scotia in bananas. Looking at the banana in their hand and then trying to estimate such a large distance is a difficult task. However, if students can reason through steps and draw on previous learning it may not be so difficult. Beginning with the concrete students may try and relate the distance of the banana to a quantifiable distance such as a meter or a foot. They could then determine the approximate number of bananas in a kilometer or a mile. It is possible that students may be able to find the driving distance or flying distance on the Internet or by using the scale on a map of Canada. If not, they would have to make a reasonable estimate for the distance. They could use reference distances such as how long it takes to travel from one city to another, such as Calgary to Winnipeg and speculate an average speed in order to determine the distance. By estimating how many times that distance is repeated across Canada, they could give an overall approximation of the amount of kilometers or miles across Canada. By using bananas as the terms of measurement, students cannot hit numbers into a calculator or use an online converter; they need to reason through the solution.

Math trail tasks are also beneficial in presenting students with non-traditional problems. For example, one of the tasks requires students to calculate the distance from a third-floor railing to the floor. If this question were in a textbook, there would not be much flexibility in making this a challenging task. Students would have to add or subtract segmented distances in order to find the answer. However, in a real-life context the task becomes much richer. There are no given dimensions or lesson titles as hints for students. The students must be creative and work together in order to solve the problem. For instance, one time a group of students tied their lanyards together, lowered the string of lanyards down from the balcony to the floor and then measured the length of the lanyards in bananas. A textbook question does not encourage this type of problem solving.

Another example of a non-traditional task is to find the volume of the banana. In classrooms students may be asked to find the volume of defined objects such as rectangular or cylindrical prisms or spheres of right cones, but are rarely asked to find the volume of irregular objects. Yet, irregular objects are common in everyday life. At first, students may be perplexed as to what to do, as they probably not aware of a formula for determining the volume of a banana. Problems in context push students to think critically and creatively. Students may try to relate the volume of the banana to something that they already know. They may use a ruler to measure 1 cm increments on the banana and then cut the banana into 1 cm cubes. Partial cubes could be combined together in order to estimate the volume. Or, in a connection to science, students could also submerge the banana in a graduated cylinder or beaker partially filled with water. The displacement of the banana could be calculated from the increased volume of the water. Problems that move beyond paper and pencil procedures to solving problems in situational contexts help develop mathematical thinking.

Conclusion

This project began as a quest for informed practice in mathematics education. My research question asked, "What are the common elements between mathematics educational research, the NCTM, and the WNCP and how can they be applied in the secondary classroom?" In the articles that were reviewed and in the common frameworks of the NCTM and WNCP the five common themes of problem solving, constructing, reasoning, connecting, and communicating were identified. These common themes were then applied to three topics in a curricular framework as illustrative examples of teaching and learning mathematics in this way.

Two extensions of the research question, beyond the scope of this project, may lead to further understanding in the teaching and learning of mathematics. They are considerations for future research and consideration for educators in applying these concepts.

Considerations for future research from this project are twofold. First, do the five common themes identified from the reviewed literature and curricular frameworks extend to subject areas outside of mathematics? Also, these five themes were applied to several curricular outcomes within the WNCP, a curricular framework that supports the development of students' mathematical thinking through flexible achievement indicators and general outcomes. How could these five themes be applied to mathematics curricular frameworks that do not support this philosophy but rather emphasis procedural skills? These two questions may lead to worthwhile research related to this topic.

They are also considerations for educators in applying these concepts in their classrooms. For some educators, the ideas presented in this project may be new and unfamiliar. Granted, educators need to start where they are situated. Change is possible, but it also takes time. At first, it will take more time in designing and planning tasks that are based in problem solving instead of isolated procedures. It will take time to become accustomed to a class in which students noisily debate the validity of mathematical constructs.

There will also be a shift. There will be in shift from teacher led discussions and the teacher viewed as the authority of what is right and what is wrong to the students accepting that responsibility. The teacher will shift from the transmitter of knowledge to a facilitator and learner with the students. There will also be a shift in the view of solutions. Solutions may not be able to be organized in a neat column on the back of a page. Solutions will become multiple and the focus will be on process and reasoning in which students arrived at a solution rather than the solution itself. These changes are necessary because the focus needs not to be on the teacher but on the student and the student's development of mathematical thinking.

Appendices

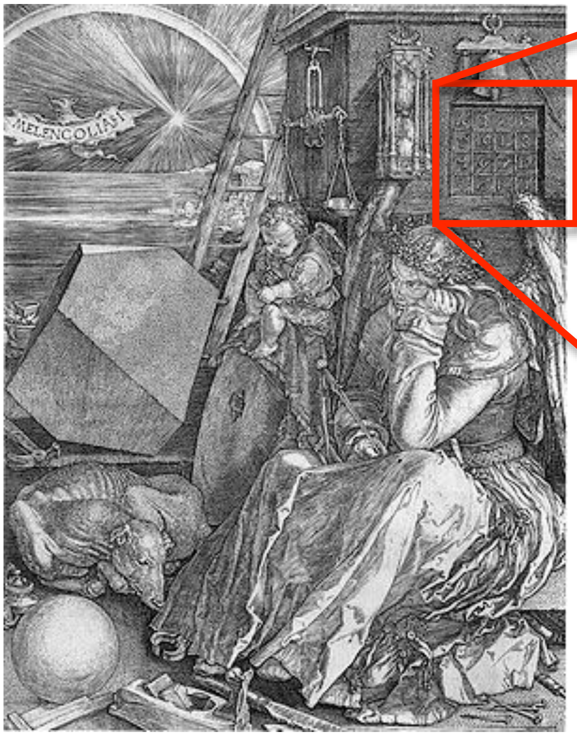
Appendix A - Magic Squares Lesson Plan

Magic squares can be found in the WNCP Curricular Framework for Grades 10-12 in the Foundations 30 pathway. Under the general outcome of logical reasoning, specific outcome number one states:

> It is expected that students will analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. *(It is intended that this outcome be integrated throughout the course by using games and puzzles such as chess, Sudoku, Nim, logic puzzles, magic squares, Kakuro and cribbage.)* (WNCP, 2008, p. 69).

Magic Squares

1. What is the sum of each row, column, and diagonal in Albrecht Dürer’s engraving in 1514 entitled “Melencolia?”



(Credit: Metropolitan Museum of Art, 2010)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

2. This sum is called the magic constant. Where else in the square is it possible to find this sum? Indicate the location(s) on the square.

3. Some people suggest that Dürer left his signature within the engraving, where might this be?

4. The numbers 1-16 are written in sequential order in a 4 x 4 grid. What translations are necessary to obtain Dürer's Square?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

5. How do you think the constant in question #1 is derived?

6. If the numbers 1-9 are used in a 3 x 3 Magic Square, what is the constant?

7. What is the constant for a 5 x 5 grid? a 6 x 6 grid? a 7 x 7 grid? a N x N grid?

8. Given the constant for a 3 x 3 grid try and find all the possible unique Magic Squares using the numbers 1-9. It may be helpful to start with the middle position. How many unique squares are there?

9. a) Try generating your own magic square. One possible way to make magic squares with an odd number of rows and columns is seen below. What pattern is followed?

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

b) Test your pattern by creating a 7 x 7 magic square in the space below.

c) What variants of this pattern can you perform and still create a magic square? Why does this work? Here are some blank squares for you to test your ideas.

10. a) Think of a 9 x 9 magic square as a normal Sudoku grid. Vary your pattern by considering each 3 x 3 square separately. When you have completed the first square in the top middle with the digits 1-9, continue using your pattern to determine the next 3 x 3 block and using the numbers 10-18. Continue in a similar manner through 73-81.

b) What patterns do you notice?

c) Why does this arrangement work?

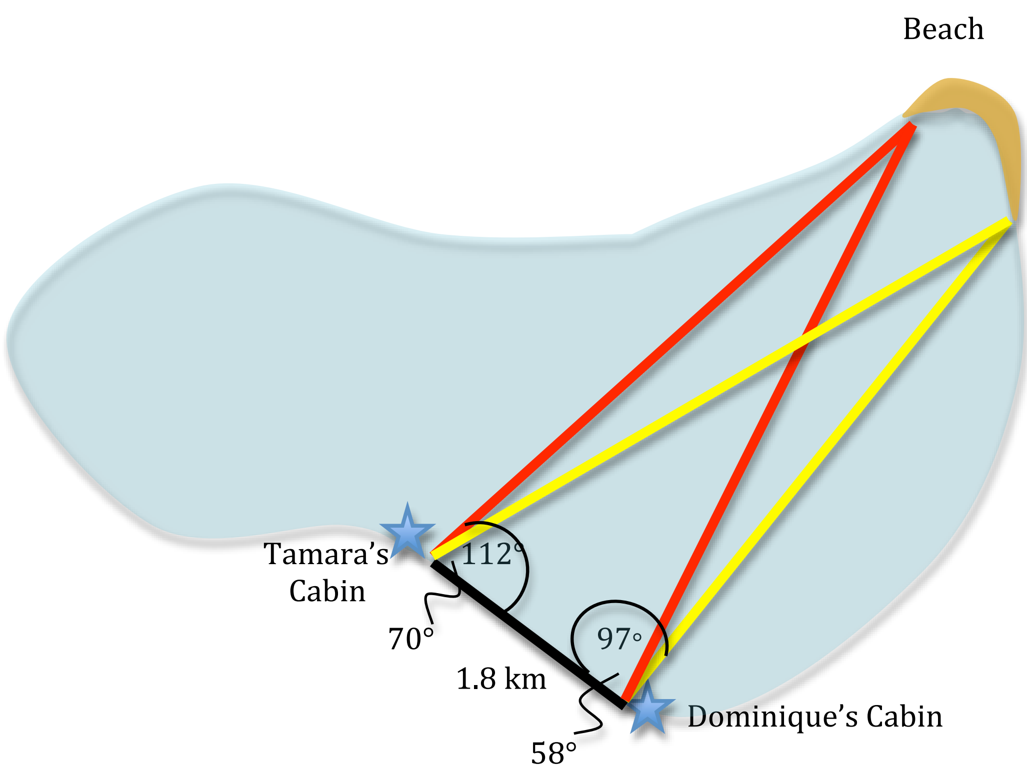
Appendix B - Sine and Cosine Law Lesson Plan

This outcome is found in all three of the pathways described in the WNCP. In Foundations 20, outcome three in geometry expects students will "solve problems that involve the cosine law and the sine law, including the ambiguous case." (WNCP, 2008, p. 59). Also, under trigonometry in Pre-calculus 29, outcome number three states that "it is expected students will solve problems that involve the cosine law and the sine law, including the ambiguous case." (WNCP, 2008, p. 81) Further, in Apprentice and Workplace Mathematics 30, under geometry, outcome number one in geometry expects students will "solve problems by using the sine law and cosine law, excluding the ambiguous case." (WNCP, 2008, p. 40).

In Foundations 20 and Pre-calculus 20 the achievement indicators show that it is expected that students not only solve problems using the sine and cosine law, but that they are also able to generate the sine law and cosine law (WNCP, 2008). Accordingly, a problem is presented at the beginning of the lesson that students will not be able to solve using trigonometric ratios in a right-angle triangle. The lesson enables students to derive the law of sines based on their previous knowledge of trigonometric ratios. The students then use this law to solve the original problem. A similar lesson framework with a different problem is also presented for the law of cosines. These lessons have been adapted from several sources including the Math Forum (Lelkes, 1997), NCTM's Illuminations resource (Godine, 2010), and Regents Prep (Roberts, 2010).

Can you canoe?

Two friends, Tamara and Dominique, at Diefenbaker Lake in Saskatchewan are wanting to canoe across the lake to a sandy beach on the other side. A road along the lake separates their two cabins. They used the odometer on their parent's mini-van to find the distance as 1.3 km. They also measured the angles from each cabin to each end of the beach using binoculars and a protractor. The angle at Tamara's cabin between the west end of the beach and Dominique's cabin is 84° while the angle at Dominique's cabin between the east end of the beach and Tamara's cabin is 86°. The other two smaller angles are shown on the diagram. Which cabin should they depart from if they want to travel the most direct route from one of the cabins to the beach? Express your answer to the nearest ten of a kilometer. The diagram is not to scale.

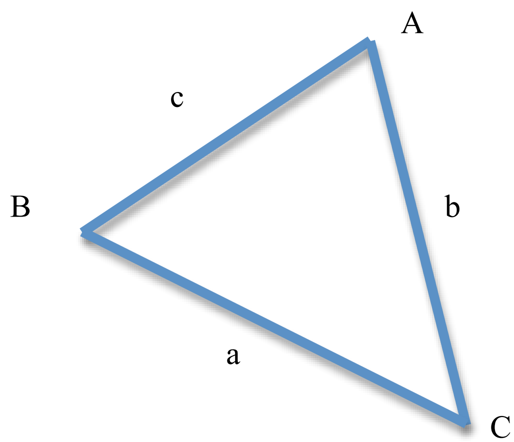


1. List the four possible routes that the girls could take as dictated by the question.

2. Looking at the value given for the angles which two routes can be immediately excluded, as they will be longer than the other two?

3. What are some possible ways of determining the distance from each cabin to the beach?

4. It is possible to find missing lengths and angles of oblique (non-right) triangles. Follow the process on a generic triangle ABC to derive the method. Begin by drawing the altitude from A. Label it h or some other letter.



5. What do you notice about the two new triangles that are formed from drawing the altitude at A?

6. How can the sides of each smaller triangle be related together? Write them as two separate formulas using the sine of < B and the sine of < C.

7. Now try and combine the two formulas into one formula. Look for the same value in each equation in order to make the equations equal to each other.

8. We are almost finished in deriving the formula to solve the beach question. Using the equation from question #7, divide both sides by b·c.

9. Use a similar method to find the relationship between angles a and b and side lengths a and b. Begin by drawing the altitude from angle C and determining relationships for sin A and sin B.

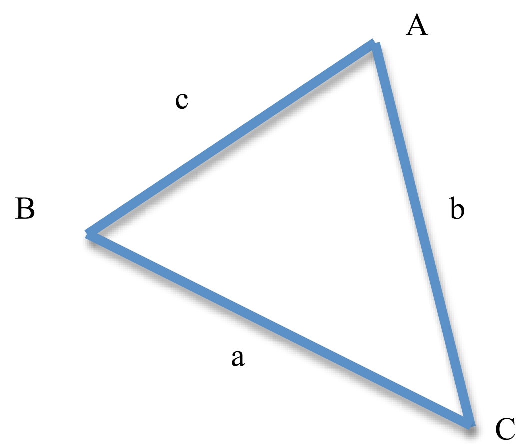
10. How can the relationships from question #8 and #9 be combined together? Combine them to make one relationship between the three sides and three angles of an oblique triangle.

11. This relationship between the three side lengths and angles of an oblique triangle is called the law of sines. Use these ratios to determine the answer to the original canoe question.

12. What is the minimum information needed in regard to side length and angles in order to use the law of sines?

13. When given a combination of three known side lengths or angles, when does the law of sines not guarantee a unique triangle?

14. Using your answer from the previous question, give two side lengths and one angle for general triangle ABC (not to scale) that has:



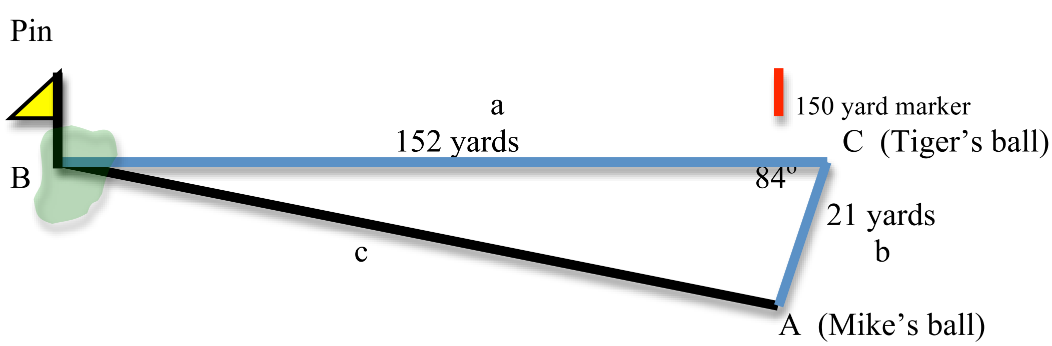
a) one unique solution

b) two possible solutions

c) no solutions

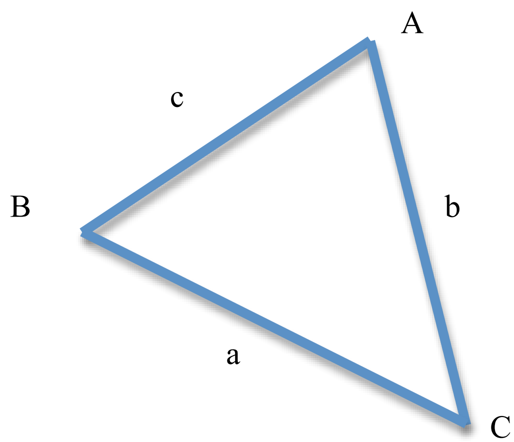
Tiger and Mike Go Golfing

Two golfers, Tiger and Mike, are playing a skins game in which points are awarded for the tee off shot that lands closest to the pin. Tiger's shot lands in the middle of the fairway, 2 yards behind a 150 yards to the pin marker while Mike's shot lands ahead of the 150 yard marker but in the rough, to the left side of the fairway. The golfers step off the distance between their two golf balls as 21 yards but obviously do not want to walk all the way to the pin to determine who is closest. If the angle at Tiger's ball between the pin and the Mike's ball is measured as 84 degrees, which golfer is closer to the hole and by how much? Calculate your answer to the nearest hundredth of a yard. The diagram is not to scale.



1. What are some possible ways to solve for side length c?

2. It is possible to find missing lengths and angles of oblique (non-right) triangles. Follow the process on a generic triangle ABC to derive the method. Begin by drawing the altitude from A. Label it h or some other letter.



3. What do you notice about the two new triangles that are formed from drawing the altitude at A? Label the bases of both smaller triangles in terms of *a* and another variable such as *x*.

4. How can the sides of each smaller triangle be related together? Write them as two separate formulas.

5. Now try and combine the two formulas into one formula. Look for the same value in each equation in order to make the equations equal to each other.

6. Next we want to solve for a side length variable such as c^2. What steps must we do in order to solve for c^2 ? Follow these steps now.

7. Last we would like to substitute the variable x for a term relating a side length (a, b, or c) with an angle (A, B, or C). How might we be able to substitute side length x with a relationship of side length b and angle C? Solve this relationship for x and substitute the value for x in the equation from question #6.

8. We now have a formula called the law of cosines that allows us to solve for side lengths and angles of oblique triangles. Use this formula to solve the original question of the two golfers and the closest ball to the pin.

9. What would be the side length of c if the angle at Tiger's ball was 90° instead of 84°? Use the law of cosines to determine the distance. What do you notice?

10. What is the minimum information needed in regard to side length and angles in order to use the law of cosines?

11. Generate the law of cosines for side lengths a and b in a similar manner as you did for side length c.

12. If the original question had given us three side lengths such as 152, 21 and 153, how could we use the law of cosines to determine the missing angles? Use the law of cosines to determine the angle at Mike's ball as formed between the hole and Tiger's ball given these three lengths.

Appendix C - Measurement Lesson Plan

Linear, surface area, and volume measurement outcomes are included in two of the WNCP pathways. In Apprentice and Workplace Mathematics 20, under measurement, outcomes one and two state:

> Solve problems that involve SI and imperial units in surface area measurements and verify the solutions" and "solve problems that involve SI and imperial units in volume and capacity measurements.
> (WNCP, 2008, p. 30)

Also, in Foundations 10, three of the outcomes under measurement are applicable. It is expected that students will:

- Solve problems that involve linear measurement, using SI and imperial unit of measure, estimation strategies, [and] measurement strategies.
- Solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including right cones, right cylinders, right prisms, right pyramids, [and] spheres.
- Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.
  (WNCP, 2008, pp. 47-48).

One possible way to meet these outcomes in a problem-solving context is to have students participate in a math trail activity. In the absence of categorized practice questions, students are forced to think critically about how to solve questions. As for the context, any school or building will have plenty of examples of objects or areas for students to measure. The following examples are taken from math trails that I developed for the University of Saskatchewan and the University of Regina. For measuring tools, I give a banana to each group of students because bananas do not have incremental marks and it requires students to convert between bananas in linear, squared, and cubed units.

Bananas Anyone?

- Your group will be in competition with other groups to get as many answers as possible within the given time constraints.
- You are allowed to tackle the questions in any order. As a group you must stay together with your leader.
- Good luck and be creative. All measurements must be in banana units and must be accurate within $\pm$ 15%.
- In order to receive full points, show your work including formulas where appropriate. Meet back here at ________.

Sample questions from a Banana Challenge at the University of Saskatchewan

________ /5 Length of the hall from doorway to doorway in the sky walk between the Arts Building and Thorvaldson.

________ /10 Volume of north elevator in the Agriculture Building.

________ /10 The amount of bananas in the flying distance between Halifax and Vancouver. Express your answer in banana scientific notation.

________ /10 The area of the 23 artistic semi-circles, with a diameter greater than ½ your banana, on the wall in Thorvaldson near Room 105.

________ /10 The distance from the top of the railing on the second floor of the Agriculture Building to the floor on the main level. No climbing or hanging over the railing.

________ /10 In the Ag Building, near Room 2E17, there is a poster entitled "Food is Important." There is a picture of a woman from Thailand wearing a red shirt in a boat. Assuming the banana second from her right is the same length as your banana, calculate the length of her boat from tip to tip in your bananas.

________ /15 The volume of the hexagonal stone base, encasing quartz and crystals in the Geology Building.

________ /15 East of the north entrance in the Agriculture Building there is a planter inscribed "Donors whose outstanding support led the way in providing to a thriving agriculture industry" Find the total surface area of the inscribed, slanted faces around the planter.

________ /15 The area of the triceratops planter in the Geology/Biology Building. You are not allowed to climb or measure across the planter. Think outside the planter.

Sample Questions from a Banana Challenge at the University of Regina

________ /5 If each horizontal wooden beam in the ceiling of the Education Pit equals one banana, how bananas are there?

________ /5 The diameter of the big blue and yellow circle around a flower and a bee (north of the Education Building).

________ /10 If there is one banana on each red seat in the Education Auditorium, including the upper wings, how many bananas are there?

________ /10 The surface area of the Coke cooler across from Security in College West.

________ /10 The volume of your banana.

________ /10 The circumference of the Earth in bananas.

_________/15 By the Theatre Department (Riddell Center) is a blue banana with wings. If one whole tile equals 1 banana squared, what is the exact area of this tiled figure?

________ /15 The volume of the whole cylindrical phone booth between the Riddell Center and the Education Building (in cubic bananas).

_________/15 The length from the 3rd floor railing to the bottom of the Admin-humanities Pit.

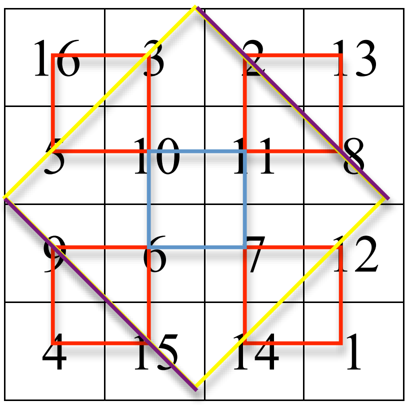
________ /15 The height of the flag pole outside of the Classroom Building.

Appendix D - Solutions

Magic squares (solutions)

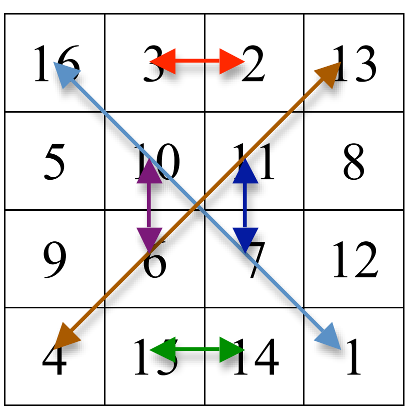
1. The constant can be found by summing a row, column or diagonal. In this case, it is 34.

2. The red squares and the blue square also add up to 34. Additionally, the two purple lines have a sum of 34 as do the two yellow lines.



3. The year the engraving made was 1514 as can be seen in the bottom middle columns. Further, some speculate that the 1 and the 4 in the bottom row correspond to Albert Dürer's initials in the German alphabet as D = 4 and A = 1.

4. If row 1, column 1 is (1,1) respectively then
(1,1) ⇔ (4,4); (1,4) ⇔(4,1); (1,2) ⇔(1,3); (4,2) ⇔(4,3); (2,2) ⇔(3,2); (2,3) ⇔(3,3)



This general pattern will work for any magic square that is a multiple of 4.

5. One possible answer. Take the sum of the numbers from 1-16 and divide by 4 since each row, column or diagonal has 4 numbers.

$(1 + 2 + 3 + 4 + \ldots 16) \div 4 = 34$

or simply

$$\left[\frac{16(16 + 1)}{2}\right] \div 4 = 34$$

6. For the numbers 1-9 in a 3 x 3 Magic Square

$$\left[\frac{9(9 + 1)}{2}\right] \div 3 = 15$$

7. For a 5 x 5 grid

$$\left[\frac{25(25 + 1)}{2}\right] \div 5 = 65$$

For a 6 x 6 grid

$$\left[\frac{64(64 + 1)}{2}\right] \div 6 = 111$$

For a 7 x 7 grid

$$\left[\frac{49(49 + 1)}{2}\right] \div 7 = 175$$

For a N x N grid

$$\left[\frac{n^2(n^2 + 1)}{2}\right] \div n$$

8. There is one unique square for a 3 x 3 square. Rotations and reflections are generally not counted as unique solutions (although this could be an interesting discussion for students).

4	9	2
3	5	7
8	1	6

9. The pattern for a magic square with an odd number of squares in each row, column or diagonal is to start with 1 in the top middle position and then place the next number

diagonally up and to the right, allowing this pattern to continue wrapping around the edges of the square. If there already is a number in this square, move directly down one square and continue the previous pattern until all the squares have been filled.

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

b) Following the same pattern as in part a) the 7 x 7 square would look like

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

9. c) Answers may vary.
Some answers may include begin one square to the right of the middle square (middle column and middle row). Use the same diagonally up and to the right rule but instead of moving down one, move two spaces to the right when a move is not possible.

Another possible answer is to begin in the middle column, one row above the center square. Use a knight's move of two down, one to the right when possible. When it is not possible, move two up.

The complete proof of all possible starting locations and moves is well beyond the scope of high school mathematics but can be found at http://www.xs4all.nl/~thospel/siamese.html (Hospel, 2002). A basic explanation is that the center square's value is the magic constant divided by the number of squares in a row, column, or diagonal. Pairs of numbers having an equal distance from the center also have the same average value. For example, in a 5 x 5 square derived by this method, 13 is the value of the center square and each same-coloured pair has 13 as its average value.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

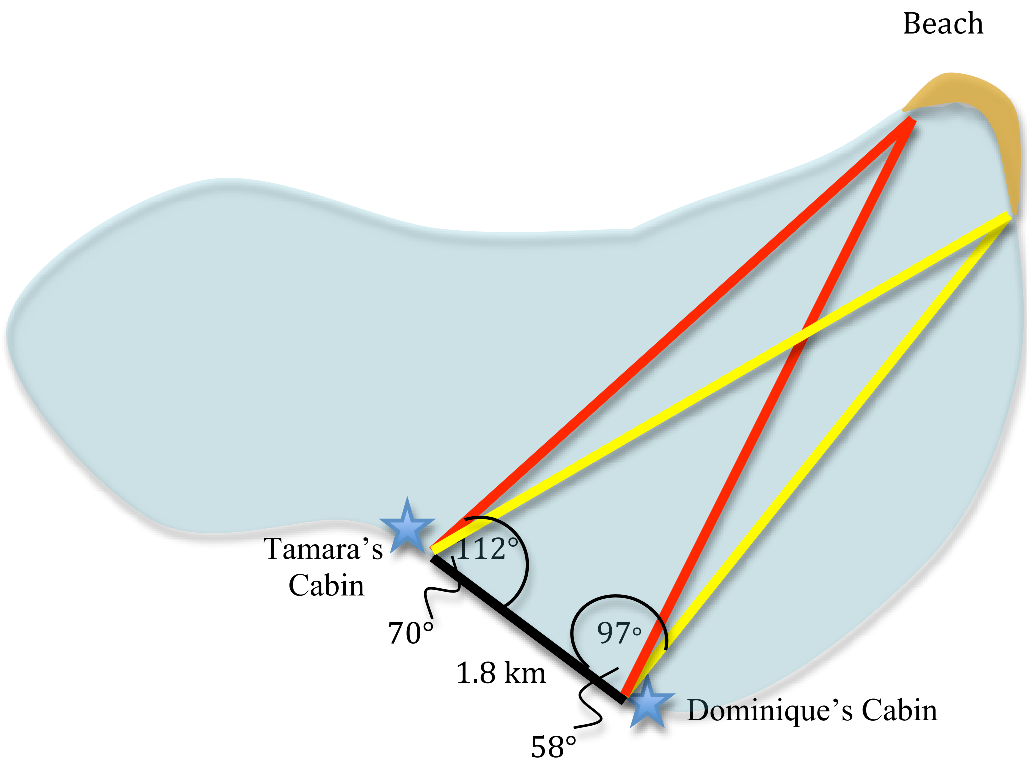
10. a)

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

b) Answers may vary but may include: this pattern generates a Magic Square as well or each 3 x 3 square is similar to the first except that 9 has been added to each value.

c) By completing a 3 x 3 square and then following the same diagonally up and right or down pattern, pairs that are equidistant from the center of 41 have an average of 41.

Can you canoe? (solutions)

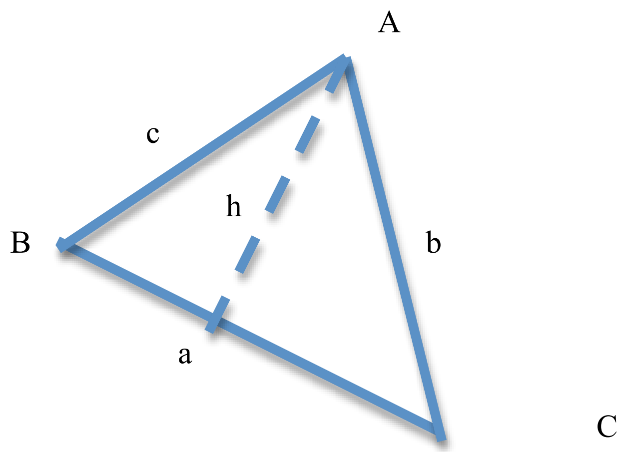


1. The four possible routes are:
a) Tamara's cabin to the west beach
b) Tamara's cabin to the east beach
c) Dominique's cabin to the west beach
d) Dominique's cabin to the east beach

2. A triangle on a two-dimensional surface can only have one angle greater than 90°. The side opposite this angle will always be the longest side as sides and opposite angles in a triangle are proportional. Accordingly, we can eliminate the distance from Dominique's cabin to the west end of the beach and the distance from Tamara's cabin to the east side of the beach as they are both opposite the largest angle of each triangle.

3. One could use trigonometric ratios if one of the angles of a formed triangle was 90°. One could also construct a diagram to scale using the two angles and side length but it would be difficult to measure the shortest distance to the beach as the beach curves.

4.



5. Each triangle is a right-angle triangle as the altitude is perpendicular to the base.

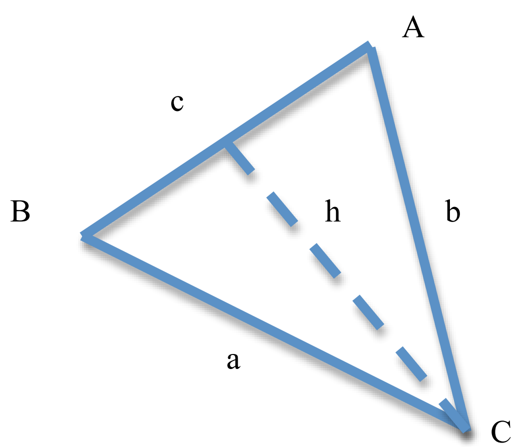
6. As the triangles contain a right angle the sides can be related by the trigonometric ratio of $\sin\theta = \frac{opp}{hyp}$ we then have $\sin B = \frac{h}{c}$ and $\sin C = \frac{h}{b}$

7. Since both equations contain h we can solve each for h. We then set the two equations equal to each other making one equation based on the transitive property.

$c \cdot \sin B = h$ and $b \cdot \sin C = h$ becomes $c \cdot \sin B = b \cdot \sin C$

8. $\frac{\sin B}{b} = \frac{\sin C}{c}$

9.



$\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}$

Once again, we solve both equations for h and set them equal to each other.

$b \cdot \sin A = h$ and $a \cdot \sin B = h$ becomes $b \cdot \sin A = a \cdot \sin B$

Dividing both sides by $a \cdot b$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

10. Using the transitive property as both $\frac{\sin A}{a}$ and $\frac{\sin C}{c}$ equal $\frac{\sin B}{b}$ it follows that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

11. In order for students to solve the original question they must find the third angle of each triangle by subtracting the sum of the other two angles by 180.

Tamara's cabin to the east shore
The third angle is 13° as 180 – (70 + 97) = 13. Using the law of sines we have

$$\frac{\text{Sin } 70}{a} = \frac{\sin 13}{1.8}$$

Solving for a we have $\frac{1.8 (\sin 70)}{\sin 13} = 7.5$ km

Dominique's cabin to the west shore
The third angle is 11° as 180 – (58 + 112) = 10. Using the law of sines we have

$$\frac{\text{Sin } 58}{b} = \frac{\sin 10}{1.8}$$

Solving for b we have $\frac{1.8 (\sin 58)}{\sin 10} = 8.8$ km

The girls should set out from Dominique's cabin and travel to the east shore, as it is the most direct distance.

12. In order to use the law of sines we need to know the measure of two angles and one side or two sides and one angle. Two angles and one side can be represented as Angle-Angle-Side (AAS) or Angle-Side-Angle (ASA). This determines a unique triangle because of congruency. Two given sides and one angle (SSA) do not necessarily determine a unique triangle. Three scenarios are possible in which there may be one unique solution, two possible solutions or no solutions.

13. In the case of two known side lengths and one known non-included angle the law of sines does not guarantee a unique triangle because SSA cannot be used to prove congruency.

14. Answers will vary. A possible solution for each question is provided.

a) One unique solution: $b = 3$, $c= 3.2$ and $< C = 28°$. This gives us 26.1° and 153.9° as possible solutions but $153.9 + 28 > 180$ so 153.9° is not possible.

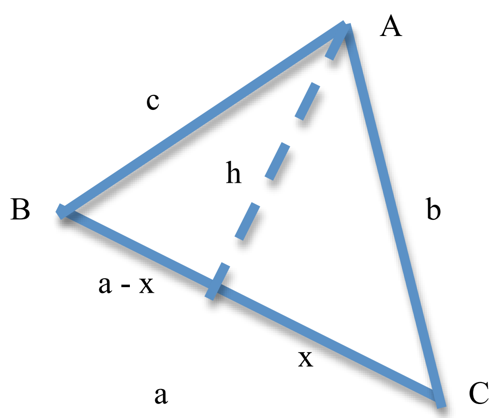
b) Two possible solutions: $b = 7$, $c = 4$ and $<C = 34°$. $<B$ can either be 78.1° or 101.9° as $78.1 + 34 < 180$ as is $101.9 + 34 < 180$.

c) No solutions: $b = 4.2$, $c = 7.6$ and $<B = 36°$. This would give us $\sin C = 1.06$, which is impossible.

Tiger and Mike go golfing (solutions)

1. If the triangle was a right angle we could use the Pythagorean Theorem to determine the missing side length as we know the lengths of the two other sides. One could use the information given and try and construct a diagram to scale and then measure the missing side.

2.



3. Each triangle is a right-angle triangle as the altitude is perpendicular to the base. Side a can be labeled as a - x and a as x represents the unknown distance of the one base and the other base can be represented as the total side length (a) minus the unknown distance (x).

4. As the triangles contain a right angle the sides can be related by the Pythagorean Theorem as $a^2 + b^2 = c^2$ or in this case $(a – x)^2 + h^2 = c^2$ and $x^2 + h^2 = b^2$.

5. Since both equations contain h^2 we can solve each for h^2. We then set the two equations equal to each other making one equation based on the transitive property.

$h^2 = c^2 - (a - x)^2$ and $h^2 = b^2 - x^2$ becomes $c^2 - (a - x)^2 = b^2 - x^2$

6. We need to expand the brackets and isolate the variable of c^2.

$c^2 - (a - x)^2 = b^2 - x^2$
$c^2 = b^2 - x^2 + (a - x)^2$
$c^2 = b^2 - x^2 + a^2 - 2ax + x^2$
$c^2 = a^2 + b^2 - 2ax$

7. As the triangle with side lengths x, b, and h contains a right angle, we can use the cosine trigonometric ratio. As $\cos\theta = \frac{adj}{hyp}$ we have $\cos C = \frac{x}{b}$

By solving for x we obtain $x = b \cdot \cos C$

8. From the previous question we had
$c^2 = a^2 + b^2 - 2ax$
$c^2 = a^2 + b^2 - 2a(b \cdot \cos C)$ by substituting for x
$c^2 = a^2 + b^2 - 2ab \cdot \cos C$ the law of cosines

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$
$c^2 = 152^2 + 21^2 - 2(152)(21)(\cos 84^o)$
$c^2 = 23545 - 667.3097095$
$c^2 = 22877.69029$
$c = 151.25$ yards

Since Mike's ball is only 151.25 yards from the hole and Tiger's ball is 152 yards to the hole, Mike's ball is closer to the hole by 0.75 yards.

9.
$c^2 = a^2 + b^2 - 2ab \cdot \cos C$
$c^2 = 152^2 + 21^2 - 2(152)(21)(\cos 90^o)$
$c^2 = 23545 - 0$
$c = 153.44$ yards

As the value of $\cos 90^o$ is 0, the second half of the equation is insignificant when using a right angle triangle. The Pythagorean theorem is a special case of the law of cosines.

10. In order to use the law of cosines, we need to have two sides and an included angle or the length of all three sides. This ensures a unique solution as Side-Angle-Side (SAS) and Side-Side-Side (SSS) defines congruency in triangles.

11.
$a^2 = b^2 + c^2 - 2bc \cdot \cos A$
$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

12. We can use the formula from #11 and solve for cos A.

$a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$\dfrac{a^2 - b^2 - c^2}{-\ 2bc} = \cos A$

$\dfrac{152^2 - 21^2 - 153^2}{-2(21)(153)} = \cos A$

$\dfrac{-746}{-6426} = \cos A$

$0.116090881 \simeq \cos A$

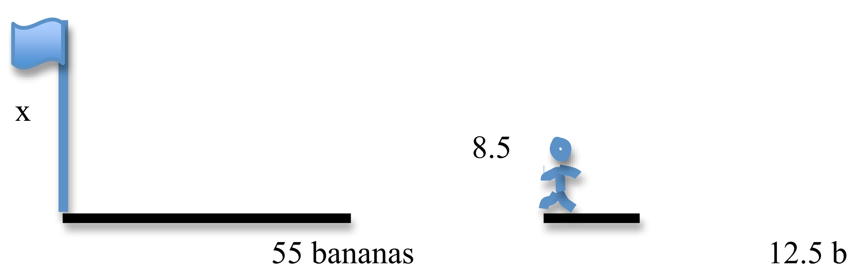
$83.33^{o} \simeq A$

The angle at Mike's ball is approximately 83.3°.

Bananas anyone? (solutions)

Possible Methods for determining the height of the flagpole

1. If the base of the flagpole is accessible to students, an easier and more familiar method may be used. Looking around the flagpole, students may notice a shadow or something. If the shadow is measurable on the ground, a simple proportion of a known height (e.g. student) and shadow can render the height of the flagpole. However, depending on weather conditions and the location of the flagpole this is not always possible.



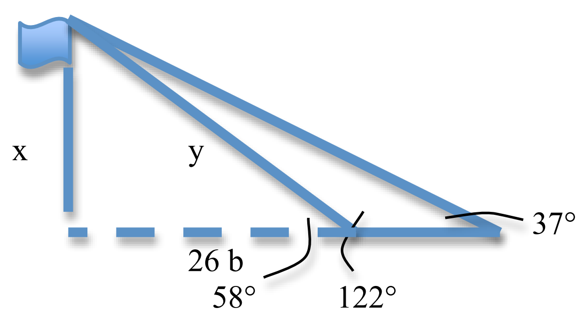
Accordingly,

$$\frac{x}{55} = \frac{8.5}{12.5}$$

x = 37.4 bananas

2. If the flagpole base is not accessible to students, the sine law can be used to calculate the height. Two separate angles of elevation can be used together with the distance between the two angles. If the angle of elevation is taken at eye level, this height needs to be added to the answer.

   For example, if the angle of elevation closer to the flagpole is 58° and the other angle of elevation is 37° with a distance of 26 bananas between the two angles, we then have:



180 – 58 = 122	Supplementary angles
180 – (37 + 122) = 21	Sum interior angles of a 2-D triangle is 180°
$\frac{\sin 21}{26} = \frac{\sin 37}{h}$	Law of sines
y $\approx$ 43.66	By solving for y
$\sin 58 = \frac{x}{43.66}$	Sine ratio in right-angle triangle
x $\approx$ 37 bananas	Solving for x

3. Another method similar to the first is for students to place a bowl of water or a mirror on the ground and back away until the tip of flagpole becomes visible in the reflection. Once again, similar triangles using the distance from the pole to the reflective surface and the distance of the person to the reflective surface

proportionate to the height of the person's eyes and the unknown height of the flag pole.

4. Using a protractor, a student can also determine the angle of elevation between his or her eyes and the top of the flagpole. The distance between the base of the flagpole and the student can be used with the angle of elevation in a sine relationship. Added together with the height of the student's visual will give the overall height.

5. A student can also take a picture of the entire flagpole with another student holding the banana along the same plane in the picture. The scale factor of the banana to the flagpole can be found by measuring the real banana and the banana in the picture.

6. One other way that is not very accurate, but can be amusing for students, as I have witnessed, is for the students to throw the banana vertically in the air. Two other students stand on either side of the flagpole a substantial distance away from the base, each with a stopwatch. If the vertical throw is successful in reaching the top of the flagpole and not proceeding farther, the two students can start the stopwatches at the peak when the velocity is 0 m/s and time the distance until the banana hits the ground. The two times can then be averaged. Applying their knowledge of quadratics, students can then use $h = -4.9\ t^2$ when h = height in meters and t = time in seconds which is derived from $h = \frac{1}{2}\ g\ t^2$ where $g = -9.8 m/s^2$.

   For example, if the average time for the banana from the height of the flag pole to the ground is 1.3 seconds then:

   $h = -4.9\ (1.3)^2$
   $h = -8.28$ meters

   We can take the absolute value of this and convert it back to bananas. By measuring a banana from tip to tip, the result may be 9 inches or 23 cm. By dividing the height in meters by the length in meters of 1 banana, we have 36 bananas as the height of the flagpole. Afterwards, the students can discuss possible errors with this method. Some examples may include not accounting for air resistance or possible horizontal velocity, the maximum height of the banana being greater than or less than the height of the flagpole, and human error in starting and stopping the stopwatch.

University of Saskatchewan Solutions

These solutions are based upon a 9-inch or 23 cm banana measuring from tip to tip.

1. 215 b
2. 772 b^3
3. 1.93 x 10^7 b
4. 24 b^2
5. 18 b
6. 16 b
7. 177 b^3
8. 60 b^2
9. 757 b^2

University of Regina Solutions

These solutions are based upon an 8 ¼-inch or 21 cm banana measuring from tip to tip.

1. 96 b
2. 17 b
3. 799 b
4. 228 b^2
5. Answers will vary.
6. 191,428,571 b
7. 4.375 b
8. 32 b^3
9. 50 b
10. 130 b

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